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LEON PESOTCHINSKY

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Remarks on BIB Designs with Repeated Blocks

by L. Pesotchinsky

1. Introduction and Summary.

Let V denote the set of v elements $1, 2, \dots, v$. A balanced incomplete block design with parameters v, b, r, k, λ, b^* denoted further as $\text{BIBD}(v, b, r, k, \lambda | b^*)$, is such a set of b elements, referred to as blocks, that:

- 1) each block contains exactly k elements of V ,
- 2) each element of V occurs in exactly r blocks and each pair of distinct elements of V appears together in exactly λ blocks,
- 3) there are exactly b^* distinct blocks among all b blocks of BIBD.

If $b^* < b$ then such design is called BIBD with repeated blocks.

The construction of designs with repeated blocks has interesting applications in experimental designing, controlled sampling and some other fields. As the examples we can consider two settings. At first, suppose that an industrial process depends on v controlled factors x_1, x_2, \dots, x_v and also on some number of unknown and uncontrolled parameters, the situation common in industrial conditions when x_1, x_2, \dots, x_v are technological parameters and the others are conditioned by variation in raw materials and so on. Then we can "tune-up" the process by changing levels of controlled parameters with accordance to some experimental design and after the data analysis find the optimal levels of x_i 's, $1 \leq i \leq v$, corresponding to the set of unknown uncontrolled variables. If the experimental scheme is once established, then we can manage to perform an "up to the moment tune-up" under the condition that the variation of unknown variables does not occur in relatively small intervals.

From a practical point of view it is too risky to variate all v technological parameters simultaneously because we can reduce the effectiveness of the process by the choice of points too remote from the optimal one established on the previous step. As usual it is sometimes impossible to try certain combinations of x_i 's for the technological reasons. The solution for such a setting could be in choosing "experimental units" of k factors from v and including in a set of such units only those which can be realized practically. The structure of the design inside the units can be arbitrary, depending on our knowledge of the process, consideration of the time and cost conditions, etc.

It is easy to notice that the choice of a BIBD for the set of experimental units would provide the simplicity of the scheme and equal conditions for all x_i 's, as well as meet all the demands above, provided we could construct such BIBD's including "favorable" (or the most informative) blocks and excluding "unfavorable" ones.

The same reasoning can be implemented in a problem of controlled sampling, when the goal is to obtain an unbiased estimator of the population mean. Without going into details, we can mention that if the population size is v and sample size is k , then the sampling design based on BIBD with the block size k can be used. And like above it is desirable to include in the support of the sampling design favorable combinations of elements and exclude unfavorable, the latter is especially important if we have to exclude the combinations of elements with the same, say, geographical or economical conditions. It is also interesting to increase the measure of favorable combinations, that is, to copy the corresponding blocks maximal possible number of times.

In the last years the papers of Foody and Hedayat (1976), Wynn (1975), van Lint (1971), van Lint and Ryser (1972), Chakrabarti (1963) and many others were devoted to this class of problems. The properties and algorithms of construction of BIBD's with repeated blocks were studied and some bounds on b^* were obtained in the above mentioned works.

In this paper the structure of some subsets of supports, that is specified in some way sets of distinct blocks, is studied and a "conditional" inequality for obtaining lower (including sharp) bounds on b^* is considered.

Let E denote the set of distinct blocks (support) of a BIBD and E_j - the set of distinct blocks repeated j times, $j \leq \lambda$, $E = \bigcup_{j=1}^{\lambda} E_j$. Thus, if n_j is the cardinality of E_j , $n_j = |E_j|$, then

$$b^* = \sum_{j=1}^{\lambda} n_j \quad \text{and}$$

(1.1)

$$b = \sum_{j=1}^{\lambda} j n_j .$$

Among all BIBD's with the fixed values v and k we can specify those with the least possible value λ_0 of λ , denoting them as $\mathcal{J}_0 = \text{BIBD}(v, b_0, r_0, k, \lambda_0 | b_0^*)$, where b_0^* is the minimal value of b^* corresponding to b_0 .

Now it is interesting to find out what could be the values of b^* 's for BIBD's with $\lambda = t\lambda_0$ and how the sharp bounds on b_0^* could be obtained. Also from the point of view of applications the structure of E_j 's and especially of E_{λ} , that is of the set with the blocks copied maximal number of times, is of some interest.

In section 2 we consider the case of $\lambda_0 = 1$. The structure of E_λ and lower bound on b^* (if $b^* - b_0^* > 0$) are found. The bound obtained is sharp either for any fixed k , or if $k = 3$ for arbitrary v not equal to 9.

In section 3 an inequality for lower bound of b^* depending on cardinality of E_λ is obtained and some examples are given, including the sharp bound $b_0^* \geq 22$ in case of $v = 8$, $k = 3$, the corresponding design was constructed by Foody and Hedayat (1976).

2. Designs with Repeated Blocks and $\lambda_0 = 1$.

Suppose that a $\text{BIBD}(v, b, r, k, \lambda | b^*) = \mathfrak{J}$ is not merely a λ -repetition of a $\text{BIBD } \mathfrak{J}_0$ with $\lambda_0 = 1$, that is $b^* > b_0 = b_0^* = \frac{v(v-1)}{k(k-1)}$. We can find now the lower bound for $b^* - b_0$.

Theorem 2.1. Under the above condition $b^* - b_0 \geq 2(k-1)$. For the proof assume that there exist two elements α and β such that their pair inclusion occurs in some block of E_λ , and in $E' = E_1 \cup E_2 \cup \dots \cup E_{\lambda-1}$ exist blocks with α and β . Then assume that there exists such element i that pair inclusions αi and βi belong to E' . Then the minimal number of distinct blocks in E' with these inclusions is equal to 4 iff they are as follows:

$$\begin{array}{ll} \alpha i & i_3 \dots i_k & \alpha i & j_3 \dots j_k \\ \beta i & i_3 \dots i_k & \beta i & j_3 \dots j_k \end{array}$$

In the same way, considering inclusions α_{i_3} and β_{i_3} we come to the conclusion that the minimal number of distinct blocks in E' containing α and β is equal to $4(k-1)$. This number corresponds to the situation with even λ , iff $E' = E_{\lambda/2}$ and the structure of $E_{\lambda/2}$ is represented by four matrices $[k-1 \times k]$:

$$(\alpha|M), (\alpha|M^*), (\beta|M), (\beta|M^*)$$

where M, M^* are matrices $[k-1 \times k-1]$ with the same distinct $(k-1)^2$ elements, any any two rows of M and M^* have exactly one common element (e.g. M^* can be the transpose of M).

Now, if we can not find such α and β as assumed above, then either $E_\lambda = \emptyset$ and it means that any pair inclusion belongs to at least two blocks, thus $b^* \geq 2b_0$, or E' is itself a BIBD \mathcal{J}' with $v = v', r = r', b = b'$ and k, λ as in \mathcal{J} . For the number of blocks n_λ in E_λ we have

$$n_\lambda = \frac{v(v-1) - v'(v'-1)}{k(k-1)}$$

and it implies that for \mathcal{J}' $\lambda_0 = 1$. Then

$$\frac{v'(v'-1)}{k(k-1)} = b'_0 \geq v',$$

thus

$$v'-1 \geq k(k-1)$$

and

$$b'^* \geq 2b'_0 \geq 2k(k-1)+2 > 4(k-1)$$

$(b^* \geq 2b_0$ because any pair of E' belongs to at least two blocks, otherwise there is a block repeated λ times, contradicting the fact $E' \cap E_\lambda = \emptyset$). Also if such α and β exist but we can not find i that α_i and β_i both belong to E' , then we can consider separately sets of blocks with α and β , say E'_α and E'_β , $E'_\alpha, E'_\beta \subset E'$, $E'_\alpha \cap E'_\beta = \emptyset$ and so on.

To complete the proof we can write

$$b^* = n_\lambda + \sum_{j=1}^{\lambda-1} n_j \geq n_\lambda + 4(k-1)$$

and

$$b_0 = n_\lambda + \frac{1}{2} \sum_{j=1}^{\lambda-1} n_j ,$$

(once again, the latter because any pair of elements which occur in E' appears at least in two distinct blocks). Both these inequalities imply

$$b^* - b_0 \geq 2(k-1) .$$

Note: In the same manner as above we can show the potential existence of the designs with

$$b^* = b_0 + 2(k-1) + 2p ,$$

where

$$0 \leq p \leq r-k+1 = \frac{v-1}{k-1} - (k-1) .$$

The existence of such designs is proved below.

Examples.

2.1. Let us consider a permutation of two elements α_1 and α_2 in a BIBD $(v, b_0, r_0, k, 1 | b_0) = \mathcal{J}_0$. By this method we obtain BIBD \mathcal{J}'_0 , \mathcal{J}_0 and \mathcal{J}'_0 have exactly $b_0 - 2(r_0-1)$ common blocks, and thus a

BIBD $\mathcal{J} = \mathcal{J}_0 \cup \mathcal{J}'_0$ has $b^* = b_0 + 2(r_0 - 1)$ distinct blocks. Now, if $r_0 = k + p$, we have the design as in theorem 2.1 above. The lower bound is reached for $r_0 = k$, that is for symmetric BIBD with $v = k(k-1)+1$.

2.2. For $k=3$ the symmetric BIBD corresponds to $v = 7$, and since any BIBD with $k=3$ and $v \geq 15$ contains at least one subsystem of blocks isomorphic to $\mathcal{J}_0 = \text{BIBD } (7, 7, 3, 3, 1/7)$ the bound can be reached also for any $v \geq 15$. If $v = 13$ and $k = 3$, we can use nonisomorphic designs \mathcal{J}_0 and \mathcal{J}'_0 as presented by Hall (1967, p. 237) to obtain the same result, and for the exclusive case of $v = 9$ we can prove that $b^* - b_0 \geq 6$. Really, if $b^* - b_0 = 4$ in this case, then the structure of E' is as in theorem 2.1 and only 6 elements are engaged in E' . But then in 8 blocks of E_λ we should have occurrence of each of the rest 3 elements four times, which can be done only in 10 blocks. Thus the construction of a BIBD with $b^* = b_0 + 4$ is impossible in this case. On the other hand $b^* = b_0 + 6$ can be reached as in example 2.1) above.

In the next part of the section we will find the conditions under which a set E_λ of blocks is a part of support of a BIBD.

Lemma 2.1. Let \mathcal{J} be a BIBD with $\lambda > 1$ and the support of \mathcal{J} consists of E_λ and $E' = \bigcup_{j=1}^{\lambda-1} E_j$. Suppose there exists a BIBD \mathcal{J}_0 with $\lambda_0 = 1$ which support contains E_λ . Then exists a BIBD \mathcal{J}_{0i} with $\lambda_0 = 1$ with the support containing $E_\lambda \cup a_i$, where a_i is an arbitrary block of E' .

For the proof let us denote the incidence submatrix corresponding to E' as $A_1 = [n' \times v]$, $\lambda n_\lambda + n' = b_0 \lambda$.

Due to the condition of the lemma there exists such a set of blocks E^* that the matrix A^* corresponding to it satisfies the equation

$$B_1 = A_1^T A_1 = \lambda A^{*T} A^* = \lambda B^*.$$

Suppose, that we can not construct E^* with A^* having a_1 as its first row. Then for any set of blocks $b_2, b_3, \dots, b_{n'/\lambda}$

$$B_1 - a_1^T a_1 - \sum_{j=2}^{n'/\lambda} b_j \neq (\lambda-1)B^*,$$

it means that we can not construct E^* without a_1 , contradicting the assumption.

The lemma enables us to prove the main result of the section:

Theorem 2.2. A set E_λ of blocks copied λ times is a part of support of a BIBD($v, b, r, k, \lambda | b^*$) iff it is a part of support of a BIBD($v, b_0, r_0, k, 1 | b_0$).

Proof. We can use the induction with respect to the parameter $n_\lambda = |E_\lambda|$. The basic statement for $n_\lambda = 1$ or 2 is evident, and suppose the statement of the theorem is valid for $n_\lambda = n$. Then, if $n_\lambda = n+1$, we can construct using the assumption for the set of blocks $E_\lambda \setminus a_1$ ($a_1 \in E_\lambda$) such set of blocks E^* that $\{E_\lambda \setminus a_1\} \cup E^*$ is the support of a BIBD with $\lambda = 1$. Then we can use lemma 2.1 with block a_1 added to E' λ times to prove, that there exists a set of blocks E^* containing a_1 . Thus the proof is accomplished.

Examples.

2.3. Let us consider a BIBD with $v = 7, k = 3$. Since any two blocks of such design with $\lambda_0 = 1$ ($b_0 = 7$) have one common element, it follows

from theorem 2.1 that E_λ can consist of 1 or 3 blocks, because any two blocks determine the third with the common element and four blocks determine the other three. On the other hand it means that $b^* \geq 11$ if $n_\lambda < 7$, because then $n_\lambda \leq 3$.

2.4. For the same situation the fact $n_\lambda = 1$ or 3 implies that if $b^* = 12$, then $n_\lambda = 3$ (else $b^* \geq 13$), and it means that all 7 elements belong to E_λ . The number of blocks 9 in E' multiplied by 3 (the number of pairs in each block) is the total number of pairs in E' , and since we must have 12 distinct pairs in E' we can write

$$(2.1) \quad \sum_{j=2}^{\lambda} x_j = 12, \quad \sum_{j=2}^{\lambda} j x_j = 9 \cdot 3,$$

where x_j is the number of pairs which belong to j distinct blocks.

(2.1) implies

$$(2.2) \quad \sum_{j=3}^{\lambda} (j-2)x_j = 3,$$

but with any pair of elements we can have at most 2 blocks, because the elements of such pairs already belong to E_λ with in total 3 other elements, thus $x_j = 0$ for $j \geq 3$ contradicting (2.2). So the construction of BIBD $(7, b, r, k, \lambda | 12)$ is impossible. The reasoning above can be illustrated by the following construction: without loss of generality we can have $E_\lambda = 123, 145, 167$; then with any pair, say 24, we can have in E' only 2 blocks (246 and 247), thus $x_j = 0$ for $j \geq 3$. In the same way we can prove that a BIBD with $b^* = b_0 + 2(k-1)+p$ does not exist for $p < k-1$.

For the values of $\lambda_0 \geq 2$ we can not expect meaningful results characterizing the properties of different sets $E_j \in E$ with respect to the design \mathcal{D}_0 with $\lambda = \lambda_0 \geq 2$, because the latter itself can have repeated blocks. The study of supports of such designs, namely with $\lambda = \lambda_0 \geq 2$, could be simplified by constructing the "minimal" design \mathcal{D}_0^* with the least size of support b_0^* . This problem is difficult itself and a method for evaluating b_0^* is considered in the next section.

3. The Conditional Bounds for the Support Size.

We will now use the information about the size of E_λ to obtain some bounds on b^* . As our sources of such information we can use either theorem 2.2 (for the case of $\lambda_0 = 1$), or some auxiliary facts, such as e.g. theorem 10.2.2 from van Lint (1974, p. 100). The latter states that if blocks b_1 and b_2 occur e_1 and e_2 times in E and λ_{12} is the number of objects common to both of them, then

$$(3.1) \quad \left(\frac{r}{e_1} - k\right)\left(\frac{r}{e_2} - k\right) \geq \left(\frac{\lambda k - r\lambda_{12}}{r - \lambda}\right)^2.$$

In particular, if $e_1 = e_2 = \lambda$, then $\lambda_{12} = 0$ or 1 and since $r = \frac{\lambda(v-1)}{k-1}$ we have from (3.1)

$$(3.2) \quad \left(\frac{v-1}{k-1} - k\right)^2 \geq \frac{k^2(k-1)^2}{(v-k)^2} \quad (\text{if } \lambda_{12} = 0)$$

or

$$\left(\frac{v-1}{k-1} - k\right)^2 \geq \frac{(k^2 - k - v + 1)^2}{(v-k)^2} \quad (\text{if } \lambda_{12} = 1).$$

It follows from the first inequality of (3.2) that $v \geq k^2$, so with $v < k^2$ we can not have disjoint blocks in E_λ . If $v = 8$ and $k = 3$ we can prove the stronger fact.

Lemma 3.1. For the BIBD $(8, 56 \frac{\lambda}{6}, 21 \frac{\lambda}{3}, 3, \lambda | b^*)$

$$n_\lambda = |E_\lambda| \leq 1.$$

Since here $v < k^2$ we have to prove only that two joint blocks can not be included in E_λ (obviously we can not have in E_λ three of them).

Suppose that two such blocks, say 123 and 145 are in E_λ . Then it follows that three blocks with element 1 and other elements must be included in $E_{\lambda/2}$:

$$167, 168, 178 \in E_{\lambda/2}.$$

Then we can write, denoting the number of copies of block ijk as e_{ijk} ,

$$\sum_{i=2}^5 e_{i\alpha\beta} = \lambda/2, \text{ where } \alpha\beta = 67, 68 \text{ or } 78$$

(3.3) and

$$\sum_{\alpha=6}^8 e_{ij\alpha} = \lambda, \text{ where } ij = 24, 25, 34, \text{ or } 35.$$

For the number of pair occurrences of the elements 2, 3, 4, 5 with the elements $\alpha = 6, 7, 8$ we have

$$(3.4) \quad 2 \sum e_{ij\alpha} + \sum_{\substack{i=2 \\ \beta \neq \alpha}}^5 e_{i\alpha\beta} = 4\lambda, \text{ where } 6 \leq \beta \leq 8 \text{ and } ij$$

as in (3.3). Taking the sum over $\alpha = 6, 7, 8$ we obtain from (3.3) and (3.4):

$$2 \cdot 4\lambda + 2 \cdot \frac{3\lambda}{2} = 12\lambda .$$

The contradiction proves that two joint blocks, as well as disjoint ones, can not be included in E_λ .

From the equation (1.1) we can find that for even λ

$$(3.5a) \quad \frac{\lambda}{2} (b^* - n_\lambda) = b - \lambda n_\lambda + \sum_{k=1}^{\lambda/2-1} \alpha_k$$

where $\alpha_k = n_1 + \dots + n_k - n_{\lambda-1} - \dots - n_{\lambda-k}$ and for odd λ

$$(3.5b) \quad \frac{\lambda}{2} (b^* - n_\lambda) = b - \lambda n_\lambda + \frac{1}{2} \left\{ 2 \sum_{k=1}^{\frac{\lambda-3}{2}} \alpha_k + \frac{\alpha_{\lambda-1}}{2} \right\} .$$

The values α_k are non-negative, because any pair which occurs in $E_{\lambda-k}$ must appear at least once (in E_k) or more (in other subsets E_j with $j < k$). Moreover, if $\alpha_k = 0$, then $\alpha_i = 0$ for all $i \leq k$. Then, if the value n_λ is the maximal one, we must have $\alpha_k > 0$ for all k , because otherwise the same design could be obtained by copying all the blocks of $\bigcup_{i=1}^k E_{\lambda-i}$ λ times and deleting the blocks of $\bigcup_{i=1}^k E_i$, thus increasing the value of n_λ .

Hence in general case we have from (3.5)

$$(3.6a) \quad b^* \geq \frac{2v(v-1)}{k(k-1)} - n_\lambda$$

and if n_λ is the maximal one,

$$(3.6b) \quad b^* \geq \frac{2v(v-1)}{k(k-1)} - n_\lambda + 1 .$$

For $v = 8, k = 3$ (3.6b) gives us

$$b^* \geq 19 ,$$

and for $v = 7, k = 3$ and $b^* > 7$ we can obtain once again

$$b^* \geq 11 \quad (\text{since } n_\lambda \leq 3) .$$

We can also use the fact that

$$(3.7) \quad \sum_{j > \lambda/2} n_j \leq \left[\frac{v}{k} \left[\frac{v-1}{k-1} \right] \right] ,$$

where $[x]$ is an integer part of x . Thus for $v = 8, k = 3$ we have

$$\sum_{j > \lambda/2} n_j \leq 8 .$$

((3.7) simply states that any pair, which appears in one of the sets E_j , $j > \lambda/2$, can not occur in any other of these sets. So $\left[\frac{v-1}{k-1} \right]$ is the maximal number of blocks with the fixed element which could belong to

$$\bigcup_{j > \lambda/2} E_j).$$

More sharp bounds on b^* could be obtained if the values α_k were properly estimated. This is done below for the case of $v = 8, k = 3$ and $\lambda_0 = 6$.

A BIBD with $b_0^* = 22$ was constructed recently in the paper of Foody and Hedayat (1976), and here we will prove that 22 is the sharp lower bound for b_0^* .

We will consider separately different values of the sum $n_6 + n_5 + n_4 = t \leq 8$, and we also need the following fact:

Lemma 3.2. Let us consider a set $E_{\lambda/2}$ from a support of $\text{BIBD}(8, 56 \frac{\lambda}{6}, 21 \frac{\lambda}{6}, 3, \lambda | b^*)$. Then at least one pair of each block of $E_{\lambda/2}$ does not occur in two blocks of $E_{\lambda/2}$.

For the proof suppose that all three pairs of one block, say 123 , belong to $E_{\lambda/2}$, it means that three blocks $12\alpha, 13\beta, 23\gamma$ also belong to $E_{\lambda/2}$.

Each of the elements $1, 2, 3$ has to appear in $\frac{7\lambda}{2} - \frac{3\lambda}{2} = 2\lambda$ other blocks of BIBD , and we have to consider now three possibilities, corresponding to the cases when all of α, β, γ are distinct and two or three of them are the same. If e.g. $\alpha = \beta \neq \gamma$ then α has to be in $\lambda/2$ blocks with each of the elements $2, 3$ and hence in $\frac{7\lambda}{2} - \lambda - 2 \cdot \frac{\lambda}{2} = \frac{3\lambda}{2}$ blocks without $1, 2, 3$. But we have $4 \cdot \frac{\lambda}{2} + 3 \cdot 2\lambda = 8\lambda$ blocks with $1, 2, 3$ in BIBD so the total number of blocks exceeds or is equal to $8\lambda + \frac{3\lambda}{2} = \frac{19\lambda}{2} > 56 \frac{\lambda}{6}$ which is impossible.

The other cases can be treated in the same way.

In particular, it follows from the lemma that if $v_{\lambda/2}$ and $\mu_{\lambda/2}$ denote respectively the numbers of pairs occurring in $E_{\lambda/2}$ λ and $\lambda/2$ times, then $v_{\lambda/2} \leq n_{\lambda/2}$ and since $2v_{\lambda/2} + \mu_{\lambda/2} = 3n_{\lambda/2}$,

$$(3.8) \quad \mu_{\lambda/2} \geq n_{\lambda/2} \geq v_{\lambda/2}.$$

Now, if $t = 8$, then 4 pairs which do not occur in $E_{456} = E_4 \cup E_5 \cup E_6$, do not have common elements. It means that they can not belong to one block and thus $n_3 = 0$, $\sum_{\alpha\beta, i} e_{i\alpha\beta} = 2^4$, where $\alpha\beta$ are those 4 pairs and $i \neq \alpha, \beta$. So $6n_6 + 5n_5 + 4n_4 = 32$ and thus $n_6 = n_5 = 0$, $n_4 = 8$. Since we can not have $n_i = 0$ for all odd $i = 1, 3, 5$ (otherwise it would

be possible to construct a BIBD with $v = 8$, $k = 3$ and $\lambda = 3$), we must have $n_1 > 0$; the pairs of E_1 must be there an even number of times, so n_1 is even and, moreover, $n_1 \geq 4$. Now in our case $\alpha_1 \geq 4$ and $3\alpha_2 = 3(n_1 + n_2 - n_4) \geq 3 \cdot 4$, because 4 pairs $\alpha\beta$ must appear at least in three different blocks each, so from (3.5)

$$3b^* \geq 56 + 4 + 4, \text{ which yields } b^* \geq 22.$$

In the same way we can prove that if $t = 7$, then $b^* \geq 22$. The only difference is that we can have either a block in E_3 (which was in E_{456} above), or E_{456} does not contain one element (that is, represents a support of a BIBD with $v = 7$), in the latter case $b^* \geq 28$. For $t = 6$ we have to consider the possibilities of 1, 2, 3 or even 4 blocks which could be included in E_3 .

As an example we can study the case of 4 blocks, and we can easily prove that it corresponds to E_3 and E_{456} as below (the sets are unique to within equivalence):

$$E_{456} = 157, 268, 356, 378, 458, 467$$

$$E_3 = 123, 124, 134, 234.$$

It follows from lemma 3.2 that all four blocks can not be included in E_3 . Moreover, we can prove that any three of them can not be included either.

Suppose, that $123, 124, 134 \in E_3$. Then we have with elements 1, 6 and 8 the only blocks 156, 158, 167 and 178, but

$$e_{156} + e_{158} = e_{167} + e_{178} = 6 - e_{157},$$

thus the sum of numbers of pair inclusions 16 and 18 is less than 12, which is impossible.

If only two blocks, say, 123, 134 belong to E_3 , then we can find that $3\alpha_2 = 3(n_1 + n_2 - n_4 - n_5) \geq 2.4 + 3.5$ (it means that each of 4 pairs from E_3 , 12, 14, 23 and 34 must appear at least in two blocks of $E_2 \cup E_1$ and each of 5 pairs 24, 16, 18, 25, 27 - in three blocks) and $3\alpha_1 = 3(n_1 - n_5) = 4 + 2\rho$ (the right-hand part represents four pairs from E_3 which have to be in E_1 , and ρ pairs which must be doubled). It can be easily shown that $\rho > 1$, and thus $\rho \geq 4$, so from (3.5) $3(b^* - 1) \geq 50 + 4 + \frac{23}{3}$ and $b^* \geq 22$. For $n_3 \leq 1$ the result follows straight from (3.5).

For $t = 5$ we can prove that the maximal number of blocks in E_3 is five; then we have 15 pairs which occur in E_{456} , at most $5(= \nu_3)$ pairs which occur six times in E_3 and at least $5(= \mu_3)$ - which are in E_3 three times, so for $\mu_3 = \nu_3 = 5$ we have as above $3\alpha_1 = 3(n_1 - n_5) = 5 + 2\rho$, ρ here is equal to 2, 5, For $\rho = 2$ the solutions of the equations

$$\begin{cases} n_4 + n_5 + n_6 = 5 \\ n_1 - n_5 = 3 \\ n_3 = 5 \end{cases}$$

can give us $b^* = \sum n_j = 21$ if $n_6 = 1$, $n_3 = 5$, $n_2 = 8 - n_5$, $n_1 = n_5 + 3$ and $n_5 = 0, 1, 2, 3$ or 4, and we can verify that for such values of n_j BIBD can not be constructed.

The other case, $\rho=5$, leads us to the BIBD from the paper of Foody and Hedayat (1976), and if $\mu_3 > 5 > v_3$, then we can use (3.5) to prove that $b^* \geq 22$:

We have

$$3n_1 = 3n_5 + \mu_3 + 2\rho ,$$

$$3n_3 = 2v_3 + \mu_3$$

(3.9) and

$$3\alpha_2 = 3(n_1 + n_2 - n_4 - n_5) \geq 2\mu_3 + 3(13 - v_3 - \mu_3) ,$$

the latter simply states that each of μ_3 pairs must appear in $E_1 \cup E_2$ at least in two blocks and each of $28 - 15 - \mu_3 - v_3$ pairs which do not belong to $E_{456} \cup E_3$ - at least in three blocks. These equations imply

$$\alpha_1 + \alpha_2 = (n_1 - n_5) + (n_1 + n_2 - n_4 - n_5) = 13 - v_3 + \frac{2\rho}{3} \geq$$

$$9 + \frac{2\rho}{3} \text{ (since } v_3 \leq 4) ;$$

and thus, because $2\rho/3$ is an integer, from (3.5) follows $b^* \geq 22$.

At last we can consider the cases with $t \leq 4$. As in (3.9) we can write

$$3(n_1 - n_5) = \mu_3 + 2\rho$$

and

$$\alpha_1 + \alpha_2 \geq 28 - 3t - v_3 + \frac{2\rho}{3} .$$

Now, from $28-3t$ pairs each may occur in 0, 1, or 2 blocks of E_3 , and we have with the help of (3.8) $3n_3 = v_3 + (v_3 + \mu_3) \leq n_3 + (28-3t)$ and thus

$$n_3 \leq 14 - \frac{3t}{2}.$$

Then

$$\alpha_1 + \alpha_2 \geq 14 - \frac{3t}{2} + \frac{2\rho}{3}$$

and as above it implies $b^* > 22$ for all cases but one: $t = 4$ and $v_3 = \mu_3 = n_3 = 8$. For the latter we can prove on the other hand, that $\rho > 2$, thus $\rho \geq 5$ and from (3.5)

$$b^* \geq 22.$$

So the sharp lower bound for b_0^* is 22. In the mentioned above paper by Foody and Hedayat it is shown, that for any $b^* \geq 22$ there is a BIBD with the support size b^* .

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For the BIB designs with repeated blocks the properties of some subsets of support are studied as well as the use of conditional bound on the support size. If the parameters v and k are such that a BIB design with $\lambda = 1$ exists, then the results provide the lower bound for the support size not equal to the minimal one and characterize the subset of blocks copied maximal number of times.

With the help of conditional bound the sharp lower bound 22 is obtained for the BIB design with $v = 8$, $k = 3$ and $\lambda = 6$, the corresponding design constructed by Foody and Hedayat (1976).